

MORITA EQUIVALENCES FROM HIGGSING TORIC SUPERPOTENTIAL ALGEBRAS

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ABSTRACT. Let A and A' be superpotential algebras of dimer models, with A' cancellative and A non-cancellative, and suppose A' is obtained from A by contracting, or ‘Higgsing’, a set of arrows to vertices while preserving a certain associated commutative ring. A' is then a Calabi-Yau algebra and a noncommutative crepant resolution of its prime noetherian center, whereas A is not a finitely generated module over its center, often not even PI, and its center is not noetherian and often not prime. We present certain Morita equivalences that relate the representation theory of A with that of A' .

We then characterize the Azumaya locus of A in terms of the Azumaya locus of A' , and give an explicit classification of the simple A -modules parameterized by the Azumaya locus. Furthermore, we show that if the smooth and Azumaya loci of A' coincide, then the smooth and Azumaya loci of A coincide. This provides the first known class of algebras that are nonnoetherian and not finitely generated modules over their centers, with the property that their smooth and Azumaya loci coincide.

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1. INTRODUCTION

The goal of this paper is to make progress in studying the representation theory of superpotential algebras obtained from non-cancellative (i.e., inconsistent) dimer models. For brevity we will refer to superpotential algebras of dimer models as ‘dimer

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algebras' (Definition 1.1). A dimer algebra $A = kQ/I$ (or quiver Q) is *cancellative* if for all paths a, p, q we have

$$(1) \quad p \sim q \text{ whenever } pa \sim qa \neq 0 \text{ (resp. } ap \sim aq \neq 0),$$

and otherwise A is non-cancellative [MoR, Condition 4.12][D, Definition 2.5, Lemma 7.3]. Non-cancellative dimer algebras are quite complicated objects—for instance, many of them have free subalgebras—whereas cancellative dimer algebras are non-commutative crepant resolutions and Calabi-Yau algebras [Bo, Br, D, M, MoR, B].¹ To accomplish our goal, we will relate non-cancellative dimer algebras to cancellative ones that share similar structure.

There are two ingredients in this strategy. First, given a dimer Q and a subset of arrows $Q_1^* \subseteq Q_1$, we form the *contracted* or *Higgsed* quiver Q' by identifying the three paths a , $h(a)$, and $t(a)$ for each $a \in Q_1^*$. If Q' is also a dimer, then there is a k -homomorphism

$$\psi : A = kQ/I \rightarrow A' = kQ'/I'$$

of dimer algebras, called a *contraction*, which sends a path in A to the corresponding path in A' [B2, Definition 4.7]. We will consider contractions ψ where A is non-cancellative and A' is cancellative.

Second, we will require that a certain associated commutative ring is preserved under ψ . To define this ring we use the notion of an ‘impression’ of A' , which is a commutative affine k -algebra B and an algebra monomorphism $\tau : A' \hookrightarrow M_{|Q'_0|}(B)$ with certain properties (Definition 1.3). By [B2, Proposition 4.5], every cancellative dimer algebra admits an impression where B is a polynomial ring. For a path $p \in A'$, denote by $\bar{\tau}(p) \in B$ the single non-zero matrix entry of $\tau(p)$. We define the associated commutative rings of A and A' to be

$$(2) \quad S := k[\cup_{i \in Q_0} \bar{\tau}\psi(e_i A e_i)] \quad \text{and} \quad S' := k[\cup_{i \in Q'_0} \bar{\tau}(e_i A' e_i)].$$

We will require that $S = S'$.

Since A' is cancellative, its center Z' , which is a 3-dimensional normal toric Gorenstein singularity [Br], is isomorphic to S' . Furthermore, the center Z of A modulo its nilradical is isomorphic to a nonnoetherian subalgebra R of S [B2, Theorem 4.15]:

$$(3) \quad Z/\text{Nil}(Z) \cong R := k[\cap_{i \in Q_0} \bar{\tau}\psi(e_i A e_i)] \subset S = S' \cong Z'.$$

In fact, R is *depicted* by S (Definition 1.2). $\text{Max } R$ may therefore be viewed as the algebraic variety $\text{Max } S$, but containing a positive dimensional subvariety that is identified as a single closed point [B2, Section 2]. In particular, the set

$$(4) \quad U := \{\mathbf{n} \in \text{Max } S \mid R_{\mathbf{n} \cap R} = S_{\mathbf{n}}\},$$

¹In Chern-Simons quiver gauge theories, cancellative and non-cancellative dimer algebras are on an equal footing [MS], while in $\mathcal{N} = 1$ quiver theories non-cancellative dimer algebras correspond to unstable but allowable physical theories.

introduced in [B2, Definition 2.2], is non-empty. This locus will play a central role throughout this paper.

For $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} := \mathfrak{q} \cap R \in \operatorname{Spec} R$, set

$$A_{\mathfrak{p}} := A \otimes_Z Z_{\mathfrak{p}} \quad \text{and} \quad A'_{\mathfrak{q}} := A' \otimes_{Z'} Z'_{\mathfrak{q}}.$$

Recall that two rings are Morita equivalent if they have equivalent module categories. In Section 2 we prove the following.

Theorem A. *The localizations $A_{\mathfrak{p}}$ and $A'_{\mathfrak{q}}$ are Morita equivalent if and only if $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$.*

Corollary B. *The noncommutative function fields $A \otimes_Z \operatorname{Frac} Z$ and $A' \otimes_{Z'} \operatorname{Frac} Z'$ are Morita equivalent, and Morita equivalent to $\operatorname{Frac} Z$ and $\operatorname{Frac} Z'$.*

Therefore A , A' , Z , and Z' are noncommutative birationally equivalent.

In Section 3 we study the Azumaya locus of A , and again find that the locus U plays an essential role.

Proposition C. *Suppose $\mathfrak{q} \in \operatorname{Spec} S$ satisfies $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$, and set $\mathfrak{p} := \mathfrak{q} \cap R$. Then the localized algebra $A_{\mathfrak{p}}$ is prime, noetherian, and a finitely generated module over its center $Z_{\mathfrak{p}}$ with PI degree $|Q_0|$.*

Let A be a finitely generated noetherian k -algebra, module-finite over its center (with the standing assumption that k is algebraically closed). Then the Azumaya locus of A is defined to be the locus

$$(5) \quad \mathcal{A} := \{\mathfrak{m} \in \operatorname{Max} Z \mid A_{\mathfrak{m}} \text{ is Azumaya over } Z_{\mathfrak{m}}\}.$$

Recall that $A_{\mathfrak{m}}$ is Azumaya over $Z_{\mathfrak{m}}$ if $A_{\mathfrak{m}}$ is a finitely generated projective $Z_{\mathfrak{m}}$ -module and the map $A_{\mathfrak{m}} \otimes_{Z_{\mathfrak{m}}} (A_{\mathfrak{m}})^{\operatorname{op}} \rightarrow \operatorname{End}_{Z_{\mathfrak{m}}}(A_{\mathfrak{m}})$, defined by $(a \otimes b) \cdot s = asb$, is an isomorphism [MR, 5.3.24]. By [BG, Proposition 3.1], if A is prime then \mathcal{A} is precisely the open dense set of $\operatorname{Max} Z$ such that for each $\mathfrak{m} \in \mathcal{A}$,

$$A_{\mathfrak{m}}/\mathfrak{m} \cong M_d(k),$$

where d is PI degree of A . In other words, the Azumaya locus consists of those points of $\operatorname{Max} Z$ whose ‘noncommutative residue fields’ have full rank.

Although the Azumaya locus has only been defined for noetherian algebras that are module-finite over their centers, we define the Azumaya locus of a general algebra A with center Z to be the subset of $\operatorname{Max} Z$ such that (5) holds. Using this definition, we show the following.

Theorem D. *The Azumaya locus \mathcal{A} of A coincides with the intersection of the Azumaya locus \mathcal{A}' of A' and the locus $U \subset \operatorname{Max} Z'$,*

$$\mathcal{A} \cong \mathcal{A}' \cap U.$$

The following corollary introduces the first known class of algebras which are not prime, noetherian, or module-finite over their centers, with the property that their Azumaya and smooth loci coincide.

Corollary E. *If the Azumaya and smooth loci of A' coincide, then the Azumaya and smooth loci of A coincide. In particular, if A' is a $Y^{p,q}$ algebra [B, Example 1.3] then the Azumaya and smooth loci of A coincide.*

We then present an explicit classification of the simple A -modules parameterized by the Azumaya locus.

Theorem F. *Suppose an A -module V_ρ sits over a point in the Azumaya locus \mathcal{A} , or equivalently, suppose V_ρ is a simple A -module of dimension 1^{Q_0} . Then there is a point $\mathfrak{b} \in \text{Max } B$ such that ρ is isomorphic to the composition*

$$A \xrightarrow{\eta} M_{|Q_0|}(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_{|Q_0|}(B/\mathfrak{b}).$$

Using this, we give a coordinate-free description of the commutative ring S and the condition $S = S'$. Denote by $\mathbb{S}(A)$ the open subvariety of $\text{Rep}_{1^{Q_0}}(A)$ consisting of simple A -modules of dimension 1^{Q_0} , and by $\overline{\mathbb{S}(A)}$ its Zariski closure.

Proposition G. *Suppose $\psi : A \rightarrow A'$ is a contraction of dimer algebras, with A non-cancellative, A' cancellative, and $S = S'$. Then*

$$S = k[\overline{\mathbb{S}(A)}]^{\text{GL}} \quad \text{and} \quad S' = k[\overline{\mathbb{S}(A')}]^{\text{GL}}.$$

In Section 4 we show that if the in-out degree of the head or tail of each contracted arrow is 1, then A contains a free subalgebra and therefore does not satisfy a polynomial identity. This is in contrast to cancellative dimer algebras, which always satisfy a polynomial identity since they are module-finite over their centers.

Theorem H. *Suppose $\psi : A \rightarrow A'$ is a contraction of dimer algebras with A cancellative, A' non-cancellative, and $S = S'$. Further suppose that for each $\delta \in Q_1^*$ the in-out degree of $h(\delta)$ or $t(\delta)$ is 1. Then A contains a free subalgebra and therefore is not PI.*

Finally in the Appendix we give a brief account of Higgsing in quiver gauge theories.

1.1. Notation and preliminary definitions. k is an algebraically closed field of characteristic zero. By module we mean left module. If an algebra is a finitely generated module over its center, then we say it is module-finite over its center. E_{ij} denotes a matrix with a 1 in the ij -th slot and zeros elsewhere.

Given a representation $\rho : A \rightarrow \text{End}_k(V)$, $V_\rho := V$ denotes the corresponding A -module defined by $av := \rho(a)v$. If $A = kQ/I$ is a quiver algebra, then by abuse of notation we also denote by ρ the corresponding vector space diagram on Q : associated to each vertex $i \in Q_0$ is a vector space k^{d_i} , and associated to each arrow $a \in Q_1$ is a linear map $\rho(a)$ from $k^{d_{t(a)}}$ to $k^{d_{h(a)}}$, the set of which satisfy the relations given by

I . The tuple $d := (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ is the dimension vector of V_ρ . If V_ρ has dimension vector $(1, \dots, 1)$, then we say V_ρ has dimension 1^{Q_0} .

Z and Z' denote the respective centers of A and A' . $\text{Max } R$ and $\text{Spec } R$ are respectively the maximal and prime ideal spectra of R . For an ideal $\mathfrak{q} \triangleleft S$, $\mathbb{V}(\mathfrak{q}) := \{\mathfrak{m} \in \text{Max } S \mid \mathfrak{m} \supseteq \mathfrak{q}\}$ denotes the closed set defined by \mathfrak{q} . Finally, if f, g are in R or S , we write $f \mid g$ if f divides g in B .

Definition 1.1. A *dimer* (or *dimer model*) is a quiver Q whose underlying graph \bar{Q} embeds into a two-dimensional real torus T^2 , such that each connected component of $T^2 \setminus \bar{Q}$ is simply connected and each cycle on the boundary of a connected component, called a *unit cycle*, is oriented and has length at least 2. A *dimer algebra* A of Q is the quiver algebra kQ/I , where

$$(6) \quad I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ.$$

For paths p, q , we will write $p \sim q$ if $p - q \in I$.

We will consider the set of arrows

$$Q_1^\dagger := \{\delta \in Q_1 \mid \rho(\delta) \neq 0 \text{ for any simple } A\text{-module } V_\rho \text{ of dimension } 1^{Q_0}\}.$$

Definition 1.2. [B2, Definitions 2.8] Suppose R is a nonnoetherian subalgebra of an affine integral domain S containing k . We say R is *depicted* by S if the set U in (4) is non-empty and the map $\text{Max } S \rightarrow \text{Max } R$, $\mathfrak{q} \mapsto \mathfrak{q} \cap R$, is surjective.

1.2. The associated commutative ring S . The following notion is needed to define the ring S , and was introduced in [B] to study a class of cancellative dimer algebras.

Definition 1.3. [B, Definition and Lemma 2.1] An *impression* (τ, B) of a k -algebra A with center Z is a commutative finitely generated k -algebra B and an algebra monomorphism $\tau : A \hookrightarrow M_d(B)$ such that (i) for each \mathfrak{q} in some open dense set $W \subseteq \text{Max } B$, the composition with the evaluation map

$$(7) \quad A \xrightarrow{\tau} M_d(B) \xrightarrow{\epsilon_{\mathfrak{q}}} M_d(B/\mathfrak{q})$$

is surjective, and (ii) the morphism $\text{Max } B \rightarrow \text{Max } \tau(Z)$, $\mathfrak{q} \mapsto \mathfrak{q}1_d \cap \tau(Z)$, is surjective.

Impressions are useful in part because they explicitly describe the center of A as a subalgebra of B [B, Lemma 2.1]. Furthermore, if A is noetherian, module-finite over its center, and B is prime, then an impression determines all simple A -modules of maximal k -dimension up to isomorphism [B, Proposition 2.5]. Specifically, if V is a simple A -module of maximal k -dimension, then there is some $\mathfrak{q} \in \text{Max } B$ such that $V \cong (B/\mathfrak{q})^d$, where $av := (\epsilon_{\mathfrak{q}}\tau)(a)v$ for $a \in A$, $v \in V$.

Definition 1.4. A *simple perfect matching* $D \subset Q_1$ is a set of arrows such that each unit cycle contains precisely one arrow in D , and $Q \setminus D$ supports a simple A -module of dimension 1^{Q_0} . We denote by $\text{Per}_s(A)$ the set of simple perfect matchings of A .

Let $A = kQ/I$ be a cancellative dimer algebra. Set

$$(8) \quad B := k[x_D \mid D \in \text{Per}_s(A)],$$

and define an algebra homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ on $Q_0 \cup Q_1$ as follows: For each $i \in Q_0$, set $\tau(e_i) := E_{ii}$, and for each $a \in Q_1$, set

$$(9) \quad \tau(a) := E_{h(a), t(a)} \prod_{a \in D \in \text{Per}_s(A)} x_D.$$

Then by [B2, Proposition 4.5] (see also [CQ, Proposition 5.3]), (τ, B) is an impression of A .

For a path p , denote by $\bar{\tau}(p)$ the single non-zero matrix entry of $\tau(p)$. We will consider contractions of dimer algebras $\psi : A \rightarrow A'$ where A is cancellative, A' is non-cancellative, and

$$S := k[\cup_{i \in Q_0} \bar{\tau}\psi(e_i A e_i)] = k[\cup_{i \in Q'_0} \bar{\tau}(e_i A' e_i)] =: S'.$$

2. MORITA EQUIVALENCES

Recall that the underlying graph of Q embeds into the real two-torus T^2 . Let $\pi : \mathbb{R}^2 \rightarrow T^2$ be the canonical projection. The covering quiver of Q is the pre-image $\tilde{Q} := \pi^{-1}(Q) \subset \mathbb{R}^2$. Fix a fundamental domain F of \tilde{Q} . For a path p in Q , we will denote by p^+ the unique path in \tilde{Q} with tail in F satisfying $\pi(p^+) = p$.

For paths p, q in Q such that $t(p^+) = t(q^+)$ and $h(p^+) = h(q^+)$, we will denote by $\mathcal{R}_{p,q}$ (resp. $\mathcal{R}_{p,q}^\circ$) the finite subquiver of \tilde{Q} bounded by p^+ and q^+ , including (resp. excluding) the arrow and vertex subpaths of p^+ and q^+ .

Denote by σ_i a unit cycle at $i \in Q_0$, and by σ the $\bar{\tau}$ -image of the unit cycles. The following lemma is well known; see for example [MoR, Lemmas 4.2, 4.4, 4.5, 4.6].

Lemma 2.1. *Suppose A is a dimer algebra. Then the following hold.*

- If σ_i, σ'_i are two unit cycles at $i \in Q_0$, then $\sigma_i \sim \sigma'_i$.
- $\sum_{i \in Q_0} \sigma_i$ is in the center of A .
- If A is cancellative and p^+ is a cycle in \tilde{Q} , then $p \sim \sigma_i^n$ for some $n \geq 0$.

Throughout $\psi : A \rightarrow A'$ denotes a contraction of dimer algebras on a set of arrows $Q_1^* \subset Q_1$, with A non-cancellative, A' cancellative, and $S = S'$ as defined in (2).

Since τ and ψ are k -homomorphisms, the composition $\bar{\eta} := \bar{\tau}\psi$ is a k -homomorphism on each corner $e_j A e_i$, $i, j \in Q_0$. We will consider the k -homomorphism

$$(10) \quad \eta : A \rightarrow M_{|Q_0|}(B)$$

defined by

$$p \mapsto \bar{\eta}(p) E_{h(p), t(p)} \quad \text{for } p \in e_j A e_i, \quad i, j \in Q_0.$$

It follows from [B2, Lemma 4.8] that η is an algebra homomorphism. However, η cannot be injective by [B2, Corollary 4.6].

Consider the lattice $\mathbb{Z}^2 \subset \tilde{Q}_0$ where the fundamental domain F of Q lies in the unit square $[0, 1) \times [0, 1)$. For $u = (u_1, u_2) \in \mathbb{Z}^2$, set

$$\tilde{Q}_0^u := \tilde{Q}_0 \cap ([u_1, u_1 + 1) \times [u_2, u_2 + 1)).$$

Denote by \mathcal{C}^u the set of cycles c in Q without cyclic proper subpaths modulo I , such that $h(c^+) \in \tilde{Q}_0^u$ (and by definition $t(c^+) \in \tilde{Q}_0^{(0,0)}$). Denote by $Q_1^u \subset Q_1$ the set of arrow subpaths of cycles in \mathcal{C}^u .

Lemma 2.2. *Let $u \in \mathbb{Z}^2$, and suppose there is a cycle in \mathcal{C}^u at each vertex of Q . Then there is a simple perfect matching D such that $D \cap Q_1^u = \emptyset$. Furthermore, for each arrow a in $Q_1 \setminus Q_1^u$ there is a simple perfect matching containing a .*

Proof. We say $c, d \in \mathcal{C}^u$ are *adjacent* if there is no cycle $t \in \mathcal{C}^u$ whose lift t^+ intersects $\mathcal{R}_{c,d}^\circ$. Consider each pair of adjacent cycles $c, d \in \mathcal{C}^u$, with d^+ to the left of c^+ in the cover \tilde{Q} .

First suppose c, d do not intersect. Since each connected component of $T^2 \setminus \bar{Q}$ is simply connected, we have the setup shown in Figure 1(i). The paths between c and d that lie in $\mathcal{R}_{c,d}$ (shown in the figure) must be arrows since there is a cycle in \mathcal{C}^u at each vertex. Denote the arrows from d to c by $D_{c,d}$.

We claim that $h(a) = t(b)$ as shown in the figure. Indeed, if not then it suffices to assume that we have the setup shown in Figure 2(a), where q is a subpath of d of positive length. Since c and d are adjacent and there is a cycle in \mathcal{C}^u at each vertex, g must be an arrow. But then $q \sim apb$, a contradiction since c and d were assumed to be adjacent. This proves our claim.

Now suppose c, d intersect at a vertex. Then we have either case (ii) or (iii) of Figure 1, where c' is a subpath of c , d' is a subpath of d , and c' and d' intersect only at the vertices $h(c') = h(d')$ and $t(c') = t(d')$. Again the paths between c and d that lie in $\mathcal{R}_{c,d}$ must be arrows since there is a cycle in \mathcal{C}^u at each vertex. In case (ii) (resp. (iii)) denote the arrows from d' to c' (resp. from c' to d') by $D_{c',d'}$.

Set

$$D := \bigcup_{c,d \text{ adjacent}} D_{c,d} \cup D_{c',d'}.$$

We claim that the oriented cycles shown in Figure 1 are unit cycles. Indeed, again consider the cycle bap shown in Figure 1(i). Since c and d are adjacent cycles, it suffices to assume we have the setup shown in Figure 2(b). Here $p = p_n \cdots p_1$, and p_1, \dots, p_n are paths. Furthermore, since c and d are adjacent and there is a cycle in \mathcal{C}^u at each vertex, the paths q, s_1, \dots, s_n must be arrows. But then

$$p = p_n \cdots p_1 \sim q\sigma_{t(p)}^{n-1},$$

a contradiction since c and d were assumed to be adjacent (and $c \in \mathcal{C}^u$ has no cyclic proper subpaths modulo I). This proves our claim.

Consequently, each unit cycle contains precisely one arrow in D . Therefore D is a perfect matching.

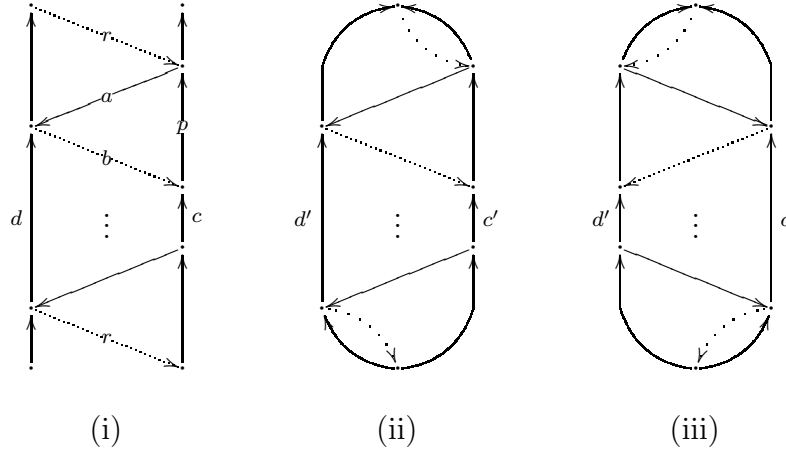


FIGURE 1. Construction of a simple perfecting matching D such that $D \cap Q_1^u = \emptyset$ in Lemma 2.2. The dotted arrows are contained in D , and the oriented cycles shown are unit cycles.

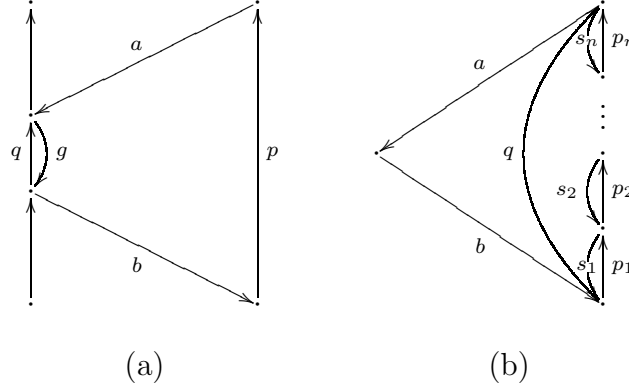


FIGURE 2. (a): Setup to show that $h(a) = t(b)$ in Figure 1(i). q is a path of positive length, and g is an arrow. (b): Setup to show that the cycle bap in Figure 1(i) is a unit cycle. p_1, \dots, p_n are paths of positive length, q, s_1, \dots, s_n are arrows, and $p = p_n \cdots p_1$.

Furthermore, for each $i, j \in Q_0$ there is path t in Q from i to j consisting of only paths denoted by solid arrows in Figure 1. Thus no arrow subpath of t is in D . Recall that a simple A -module of dimension 1^{Q_0} is characterized by the property that there is a non-annihilating path between any two vertices of Q . Therefore $Q_1 \setminus D$ supports a simple A -module of dimension 1^{Q_0} . \square

Lemma 2.3. *Let $u \in \mathbb{Z}^2$, and suppose there is a cycle in \mathcal{C}^u at each vertex. If $b, c \in \mathcal{C}^u$ are cycles at the same vertex $i \in Q_0$, then $b \sim c$.*

Proof. Since there is a cycle in \mathcal{C}^u at each vertex, there are cycles $b = b_0, b_1, \dots, b_n = c \in e_i A e_i \cap \mathcal{C}^u$ such that b_i and b_{i+1} are adjacent. Recall cases (ii) and (iii) of Figure 1. By Lemma 2.2, the paths between c'^+ and d'^+ that lie in $\mathcal{R}_{c',d'}^\circ$ are arrows, and therefore $c' \sim d'$. It follows that

$$b = b_0 \sim b_1 \sim \dots \sim b_n = c.$$

□

Lemma 2.4. *Let $u \in \mathbb{Z}^2$, and suppose there is a cycle in \mathcal{C}^u at each vertex. If $c \in \mathcal{C}^u$, then $\sigma \nmid \bar{\eta}(c)$.*

Proof. Since $c \in \mathcal{C}^u$, by Lemma 2.2 there is a simple perfect matching D such that c has no arrow subpath in D . Therefore $x_D \nmid \bar{\eta}(c)$. Thus, since $\sigma = \prod_{D \in \text{Per}_s(A)} x_D$, $\sigma \nmid \bar{\eta}(c)$. □

Lemma 2.5. *Let $u \in \mathbb{Z}^2$, and suppose there is a cycle in \mathcal{C}^u at each vertex. If $b, c \in \mathcal{C}^u$, then $\bar{\eta}(b) = \bar{\eta}(c)$.*

Proof. Let r be a path whose lift r^+ is a path from $t(c^+)$ to $t(b^+)$, and let s be a path whose lift s^+ is a path from $h(b^+)$ to $h(c^+)$. By Lemma 2.1, $\sigma^m \bar{\eta}(c) = \bar{\eta}(sbr)$ for some $m \in \mathbb{Z}$. But $\bar{\eta}(sbr) = \bar{\eta}(sr) \bar{\eta}(b) = \sigma^n \bar{\eta}(b)$ for some $n \geq 0$. Thus $\bar{\eta}(b) = \sigma^{m-n} \bar{\eta}(c)$. Therefore by Lemma 2.4, $\bar{\eta}(b) = \bar{\eta}(c)$. □

Lemma 2.6. *Suppose $g \in R$ and $\sigma \nmid g$. Then there is some $u \in \mathbb{Z}^2$ such that for each $i \in Q_0$ there is a cycle $c \in e_i A e_i \cap \mathcal{C}^u$ satisfying $\bar{\eta}(c) = g$.*

Proof. Consider cycles p, q satisfying $\bar{\eta}(p) = \bar{\eta}(q)$, and let $u, v \in \mathbb{Z}^2$ be such that $\mathcal{C}^u \ni p$ and $\mathcal{C}^v \ni q$. Suppose to the contrary that $u \neq v$.

Since $g \neq \sigma^m$ for some $m \geq 0$, p^+ and q^+ are not cycles. Thus $u \neq (0, 0)$ and $v \neq (0, 0)$. Therefore p and q must intersect at a vertex i .

Let p' and q' be the cyclic permutations of p and q at vertex i . Then $\bar{\eta}(p') = \bar{\eta}(p) = \bar{\eta}(q) = \bar{\eta}(q')$. Thus, since $\bar{\tau}$ is injective, $\psi(p') = \psi(q')$. But then there is a cycle c satisfying $t(c^+) = h(p'^+)$ and $h(c^+) = h(q'^+)$, that is contracted to a vertex. This is a contradiction by [B2, Lemma 4.10]. □

Lemma 2.7. *Suppose $g \in R$ and $\sigma \nmid g$. Then for each $\delta \in Q_1^\dagger$ there is a cycle $c \in e_{t(\delta)} A e_{t(\delta)}$ such that $\bar{\eta}(c) = g$ and $c = d\delta$ for some path d .*

Furthermore, if $b \in e_{t(\delta)} A e_{t(\delta)}$ satisfies $\bar{\eta}(b) = g$, then $b \sim d\delta$.

Proof. Since R is generated by monomials, we may assume g is a monomial. By Lemma 2.6, there is some $u \in \mathbb{Z}^2$ and cycle $b \in e_{t(\delta)} A e_{t(\delta)} \cap \mathcal{C}^u$ such that $\bar{\eta}(b) = g$. By Lemma 2.2, δ is a subpath of some cycle $c \in \mathcal{C}^u$. By possibly cyclically permuting c , we may assume $c = d\delta \in e_{t(\delta)} A e_{t(\delta)}$ for some path d . Therefore by Lemma 2.5, $\bar{\eta}(c) = \bar{\eta}(b) = g$.

Suppose $b \in e_{t(\delta)} A e_{t(\delta)}$ satisfies $\bar{\eta}(b) = g$. Since $\sigma \nmid g$, b has no cyclic proper subpaths modulo I . Thus $b \in \mathcal{C}^u$. Therefore $b \sim d\delta$ by Lemmas 2.3 and 2.6. \square

Lemma 2.8. *Consider a cycle $p \in e_i A e_i$ whose lift p^+ is a cycle in \tilde{Q} . Suppose V_ρ is an A -module of dimension 1^{Q_0} with the properties that $\rho(\sigma_i) = 0$, and for each $\ell \in Q_0$ there is a cycle $c_\ell \in e_\ell A e_\ell$ such that $\rho(c_\ell) \neq 0$. Then $\rho(p) = 0$.*

Proof. We proceed by induction on the number of vertices of \tilde{Q} in the finite subquiver $\mathcal{R}_p^\circ \subset \tilde{Q}$ bounded by p^+ . If \mathcal{R}_p° contains no vertices then p is a unit cycle, whence $p \sim \sigma_i$, and so by assumption $\rho(p) = 0$.

Now suppose \mathcal{R}_p° contains n vertices. Further suppose $\rho(q) = 0$ for any cycle q whose lift q^+ is a cycle, and which bounds a compact region whose interior contains at most $n-1$ vertices. Denote by σ_i a unit cycle at i contained in \mathcal{R}_p . Since $\rho(\sigma_i) = 0$, there is an arrow subpath a of σ_i such that $\rho(a) = 0$.

First suppose $h(a) \neq i$. By assumption there is a cycle $c \in e_{h(a)} A e_{h(a)}$ such that $\rho(c) \neq 0$.

Let $c^+ \in \pi^{-1}(c)$ be a path in \tilde{Q} that passes through the vertex $h(a)^+ \in \tilde{Q}_0$. If c^+ is a cycle contained in \mathcal{R}_p , then by induction $\rho(c) = 0$, a contradiction. Therefore c^+ is either a cycle not contained in \mathcal{R}_p , or c^+ is a non-cyclic path. If c^+ is a non-cyclic path, we may take it to have sufficient length so that its head and tail are not contained in \mathcal{R}_p . Therefore in either case, c^+ intersects p^+ in at least two vertices $j, k \in \tilde{Q}_0$.

Let r be the subpath of c^+ from j to k containing $h(a)$, and let p' be the subpath of p^+ from k to j . Since $h(a) \neq i$, $p'r$ is a cycle in \tilde{Q} that bounds a compact region which contains at most $n-1$ vertices. Thus by induction, $\rho(p'r) = 0$. But $\rho(r) \neq 0$ since $\rho(c) \neq 0$. Thus $\rho(p') = 0$ since V_ρ has dimension 1^{Q_0} . Therefore $\rho(p) = 0$.

Otherwise suppose $h(a) = i$. Since Q is non-degenerate, it contains no unit cycle of length 1. Therefore $t(a) \neq i$. We may thus apply a similar argument as above to conclude that $\rho(p) = 0$. \square

Proposition 2.9. *Suppose $p, q \in e_j A e_i$ are paths such that $p \not\sim q$, and suppose r is a path of minimal length such that $rp \sim rq \neq 0$ or $pr \sim qr \neq 0$. Then each arrow subpath of r is in Q_1^\dagger .*

Proof. Let D be a simple perfect matching, let V_ρ be a simple A -module of dimension 1^{Q_0} supported on $Q_1 \setminus D$, and suppose $rp \sim rq$.

(a) First suppose r is an arrow. Further suppose there is only one arrow a with tail at j whose lift lies in $\mathcal{R}_{p,q}$. Then by [B2, Lemma 4.12], $r = a$.

Since V_ρ is simple of dimension 1^{Q_0} , there is a path $t \in e_i A e_j$ that passes through each vertex in Q such that $\rho(t) \neq 0$. Thus $\bar{\eta}(t) \in k[\cap_{i \in Q_0} \bar{\eta}(e_i A e_i)]$. Whence $\bar{\eta}(tp) \in k[\cap_{i \in Q_0} \bar{\eta}(e_i A e_i)]$. Therefore by (3), $(tp)(tq) \sim (tq)(tp)$.

We may therefore apply the argument in [B2, part (a) of proof of Theorem 4.14] to conclude that $stp \sim stq$ for some subpath s of tq . Let $t' \in Ae_j$ be the subpath of st of minimal length such that $t'p \sim t'q$.

We claim that t' is a subpath of t . Indeed, since V_ρ is supported on $Q_1 \setminus D$, each unit cycle is represented by zero. Thus by Lemma 2.8, the lift t^+ cannot be a cycle in \tilde{Q} since $\rho(t) \neq 0$. Thus $h(t^+)$ does not lie in $\mathcal{R}_{p,q}^\circ$. But by [B2, Lemma 4.12], $h(t'^+)$ lies in $\mathcal{R}_{p,q}$. Therefore t' must be a subpath of t , proving our claim.

Since there is only one arrow a with tail at j whose lift lies in $\mathcal{R}_{p,q}^\circ$, a must be the rightmost arrow subpath of t' . Thus a is a subpath of t . But $\rho(t) \neq 0$, and therefore $\rho(a) \neq 0$.

(b) Now suppose there is more than one arrow with tail at j whose lift lies in $\mathcal{R}_{p,q}$. Consider the sets of arrows $\{b_1, \dots, b_n\} \subset e_j Q_1$ and $\{a_1, \dots, a_{n-1}\} \subset Q_1 e_j$ whose lifts lie in $\mathcal{R}_{p,q}$, ordered clockwise or counter-clockwise. Let $\{p_1, \dots, p_n\} \subset e_j Ae_i$ be the set of paths whose lifts have no cyclic proper subpaths (modulo I) and are contained in $\mathcal{R}_{p,q}$, such that for each ℓ , b_ℓ is the leftmost arrow subpath of p_ℓ . Then $p = p_1$ and $q = p_n$ (or $p = p_n$ and $q = p_1$). If $p_\ell \not\sim p_{\ell+1}$ then we may apply argument (a) with $p_\ell, p_{\ell+1}, a_\ell$ in place of p, q, a respectively, to conclude that $\rho(a_\ell) \neq 0$. Otherwise $p_\ell \sim p_{\ell+1}$, and so r cannot be a_ℓ since r has minimal length and [B2, Lemma 4.12] holds for each finite subquiver $\mathcal{R}_{p_\ell, p_{\ell+1}}$.

(c) Finally, suppose $r = r_n \cdots r_1$ with each $r_\ell \in Q_1$ and $n \geq 2$. By induction, assume the proposition holds when r has length at most $n-1$. Since r is minimal, we may apply arguments (a) and (b) with the paths $(r_{n-1} \cdots r_1)p$, $(r_{n-1} \cdots r_1)q$, r_n in place of p, q, a respectively. Thus $\rho(r_n) \neq 0$. Therefore $\rho(r) \neq 0$, and so no arrow subpath of r is in D .

The proposition follows since D was an arbitrary simple perfect matching. \square

Lemma 2.10. *If $\mathbf{n} \in U$ and $\mathbf{m} := \mathbf{n} \cap R$, then there is a (non-constant) monomial g in $R \setminus \mathbf{m}$ such that $\sigma \nmid g$.*

Proof. (a) Since S is generated by monomials and $R \subsetneq S$, there must be some monomial h in $S \setminus R$. Since $\mathbf{n} \in U$, we have $R_{\mathbf{m}} = S_{\mathbf{n}}$. Thus there are polynomials $f_1, f_2 \in R$ with $f_2 \notin \mathbf{m}$ such that $h = \frac{f_1}{f_2} \in R_{\mathbf{m}}$. Since h is a monomial and R is generated by monomials, f_1 and f_2 may be chosen to be monomials as well.

(b) Now assume to the contrary that if g is a monomial in $R \setminus \mathbf{m}$, then $\sigma \mid g$. Let p be a path of positive length in Q such that $\bar{\eta}(p) \in R$ and $\sigma \nmid \bar{\eta}(p)$. Then by assumption, $\bar{\eta}(p) \in \mathbf{m}$.

Let q be path such that $t(q^+) = h(p^+)$ and $h(q^+) = t(p^+)$. Then $(qp)^+$ is a cycle in \tilde{Q} , and so $\bar{\eta}(qp) = \sigma^n$ for some $n \geq 1$ by Lemma 2.1. Thus

$$\sigma^n = \bar{\eta}(qp) = \bar{\eta}(q)\bar{\eta}(p) \in \mathbf{m}.$$

Therefore $\sigma \in \mathbf{m}$ since \mathbf{m} is prime.

Now let $f \in R$ be a monomial such that $\sigma \mid f$, say $f = f'\sigma$ with $f' \in S$. Since σ is in \mathbf{m} , σ is also in \mathbf{n} . Therefore $f = f'\sigma \in \mathbf{n}$. But then $f \in \mathbf{n} \cap R = \mathbf{m}$.

Therefore all monomials in R are in \mathfrak{m} , a contradiction by (a). \square

Remark 2.11. By abuse of notation, we will denote by $\mathfrak{p} \in \text{Spec } R$ both a prime ideal of $R \cong Z/\text{Nil}(Z)$, and the prime ideal \mathfrak{p}' of Z containing $\text{Nil}(Z)$ such that $\mathfrak{p}' + \text{Nil}(Z) = \mathfrak{p}$. Furthermore, a maximal ideal of Z may be viewed as a maximal ideal of $R \cong Z/\text{Nil}(Z)$, and conversely. Indeed, let $\mathfrak{m} \in \text{Max } Z$. Then $\mathfrak{m} \supset \text{Nil}(Z)$, so $\mathfrak{m} + \text{Nil}(Z) = \mathfrak{m}$.

Theorem 2.12. *Suppose $\mathfrak{q} \in \text{Spec } S$ satisfies $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$, and set $\mathfrak{p} := \mathfrak{q} \cap R$. Consider paths $p, q \in e_j A e_i$ such that $\psi(p) \sim \psi(q)$. Then $p \sim q$ in the localization $A_{\mathfrak{p}}$. In particular,*

$$\text{Nil}(Z) \cdot Z_{\mathfrak{p}} = 0.$$

Proof. Fix $\mathfrak{n} \in \mathbb{V}(\mathfrak{q}) \cap U$; then \mathfrak{p} is contained in $\mathfrak{m} := \mathfrak{n} \cap R \in \text{Max } R$.

Suppose $p \not\sim q$ and $\psi(p) \sim \psi(q)$. Let r be a path of minimal length such that $rp \sim rq$. Then by Proposition 2.9, each arrow subpath of r is in Q_1^\dagger .

First suppose r is an arrow. Then by Lemma 2.10, there is a cycle $c \in e_j A e_j$ such that $\bar{\eta}(c) \in R \setminus \mathfrak{m}$ and $\sigma \nmid \bar{\eta}(c)$. Thus by Lemma 2.7, there is a path d such that $c \sim dr$. Furthermore, since $R \cong Z/\text{Nil}(Z)$, there is a central element $z \in Z$ such that $ze_j = c$. Therefore in $A_{\mathfrak{p}}$,

$$(11) \quad p - q = \frac{z}{z}(p - q) = \frac{c}{z}(p - q) \sim \frac{d}{z}r(p - q) = 0.$$

Now suppose $r = r_n \cdots r_1$ with each $r_\ell \in Q_1$ and $n \geq 2$. As above, for each ℓ there is a path $d_\ell \in e_{t(r_\ell)} A e_{h(r_\ell)}$ such that $c_\ell \sim d_\ell r_\ell$. Then $d_1 \cdots d_n r$ is a cycle at j such that $\bar{\eta}(d_1 \cdots d_n r) = s^n \in R$. But $s^n \notin \mathfrak{p}$ since \mathfrak{p} is prime and $s \notin \mathfrak{p}$. Therefore $p - q = 0$ in $A_{\mathfrak{p}}$ as in (11).

Finally, let $z \in \text{Nil}(Z)$. By [B2, Theorem 4.14], we may assume $z = p - q$, where $p, q \in e_j A e_i$ are paths satisfying $\psi(p) \sim \psi(q)$. Therefore $z = p - q = 0$ in $Z_{\mathfrak{p}}$. \square

Proposition 2.13. *Suppose $\mathfrak{q} \in \text{Spec } S$ satisfies $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$, and set $\mathfrak{p} := \mathfrak{q} \cap R$. If $\delta \in Q_1^\dagger$, then $A_{\mathfrak{p}}$ contains an element $\delta^* \in e_{t(\delta)} A_{\mathfrak{p}} e_{h(\delta)}$ satisfying*

$$\delta^* \delta = e_{t(\delta)} \quad \text{and} \quad \delta \delta^* = e_{h(\delta)}.$$

Proof. Fix $\mathfrak{n} \in \mathbb{V}(\mathfrak{q}) \cap U$; then \mathfrak{p} is contained in $\mathfrak{m} := \mathfrak{n} \cap R \in \text{Max } R$. Suppose $\delta \in Q_1^\dagger$.

By Lemma 2.10, there is a cycle $c \in e_{t(\delta)} A e_{t(\delta)}$ such that $\bar{\eta}(c) \in R \setminus \mathfrak{m}$ and $\sigma \nmid \bar{\eta}(c)$. By Lemma 2.7, there is then a path d such that $c \sim d\delta$. Furthermore, since $R \cong Z/\text{Nil}(Z)$, there is a central element $z \in Z$ such that $ze_{t(\delta)} = c$.

Consider the element

$$\delta^* := \frac{d}{z} \in A_{\mathfrak{p}}.$$

Then in $A_{\mathfrak{p}}$,

$$\delta^* \delta = \frac{d\delta}{z} = \frac{c}{z} = \frac{e_{t(\delta)} z}{z} = e_{t(\delta)} \frac{z}{z} = e_{t(\delta)},$$

and similarly $\delta \delta^* = e_{h(\delta)}$. \square

Proposition 2.14. *If V_ρ is a simple A -module of dimension 1^{Q_0} and p is a path that is contracted to a vertex, then $\rho(p) \neq 0$. Therefore $Q_1^* \subseteq Q_1^\dagger$.*

Proof. Fix a simple perfect matching $D \subset Q_1$. Let V_ρ be a simple A -module of dimension 1^{Q_0} supported on $Q_1 \setminus D$. Set $\mathfrak{m} := \text{ann}_Z(V_\rho) \in \text{Max } Z$. By Remark 2.11, \mathfrak{m} may also be viewed as a maximal ideal of $R \cong Z/\text{Nil}(Z)$.

By [B2, Theorem 4.15], $R \cong Z/\text{Nil}(Z)$ is depicted by S , so there is some $\mathfrak{n} \in \text{Max } S$ such that $\mathfrak{n} \cap R = \mathfrak{m}$. By (3), $S = S' \cong Z'$. Therefore \mathfrak{n} may be viewed as a maximal ideal of Z' satisfying $\mathfrak{n} \cap R = \mathfrak{m}$. Let \mathfrak{a} be a maximal ideal of A' containing \mathfrak{n} .

Since A' is module-finite over a noetherian central subring Z' , its primitive and maximal ideal spectra coincide. Thus there is a simple A' -module $V_{\rho'}$ such that $\text{ann}_{A'}(V_{\rho'}) = \mathfrak{a}$. In particular, $\text{ann}_{Z'}(V_{\rho'}) = \mathfrak{a} \cap Z' = \mathfrak{n}$.

Now since V_ρ is simple of dimension 1^{Q_0} , there is a path t in Q which contains each arrow in $Q_1 \setminus D$ as a subpath and satisfies $\rho(t) \neq 0$. Thus $t \in k[\cap_{i \in Q_0} \bar{\eta}(e_i A e_i)]$ since t passes through each vertex. Therefore by (3), $t \in Z e_{t(t)}$. Whence $\bar{\eta}(t) \in R$. Furthermore, since $\rho(t) \neq 0$, $\bar{\eta}(t) \notin \mathfrak{m}$. Thus $\bar{\eta}(t) \notin \mathfrak{n}$.

Therefore $\rho'(\psi(t)) \neq 0$. But then $\rho'(b) \neq 0$ for each $b \in \psi(Q_1 \setminus D) = Q'_1 \setminus \psi(D)$.

Since $\rho(a) = 0$ for $a \in D$, each unit cycle in Q is represented by zero. Thus $\sigma \in \mathfrak{m} = \mathfrak{n} \cap R$. Therefore each unit cycle in Q' vanishes under ρ' . Thus there is an arrow $b \in Q'_1$ such that $\rho'(b) = 0$. Whence $b \in \psi(D) \cap Q_1$, and in particular $\psi(D) \cap Q_1 \neq \emptyset$.

Thus for each $a \in D$, $\psi(a) \in Q'_1$ since D is a perfect matching. Therefore $Q_1^* \cap D = \emptyset$. The proposition follows since D was an arbitrary simple perfect matching. \square

We construct a new quiver \widehat{Q} from Q' as follows:

Denote by T the set of vertices $i \in Q'_0$ such that i is the ψ -image of a non-vertex path in Q . Let \widehat{Q} have vertex set Q'_0 , and arrow set Q'_1 together with two loops γ_i and γ_i^* at each vertex $i \in T$. Set

$$\widehat{A} := k\widehat{Q} / \langle I', \gamma_i \gamma_i^* - e_i, \gamma_i^* \gamma_i - e_i \mid i \in T \rangle.$$

Recall that if \mathbf{P} and \mathbf{Q} are unital idempotent rings (i.e., $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{Q}^2 = \mathbf{Q}$), then \mathbf{P} and \mathbf{Q} are Morita equivalent if and only if there is a surjective Morita context $(\mathbf{P}, \mathbf{Q}, M, N, \phi, \theta)$ [GS, Proposition 2.3]. Here ${}_P M_Q$ and ${}_Q N_P$ are bimodules and

$$\phi : N \otimes_P M \rightarrow \mathbf{Q} \quad \text{and} \quad \theta : M \otimes_Q N \rightarrow \mathbf{P}$$

are surjective bi-module homomorphisms satisfying the associativity conditions

$$\phi(n \otimes m)n' = n\theta(m \otimes n') \quad \text{and} \quad \theta(m \otimes n)m' = m\phi(n \otimes m')$$

for each $m, m' \in M$ and $n, n' \in N$.

Theorem 2.15. *Let $\psi : A \rightarrow A'$ be a contraction of dimer algebras with A non-cancellative and A' cancellative, and $S = S'$. Then the localizations $A_{\mathfrak{p}}$ and $A'_{\mathfrak{q}}$ are Morita equivalent if and only if $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$.*

Proof. For ease of notation, first suppose only one arrow δ in Q is contracted, and set

$$P := A_p = A \otimes_Z Z_p \quad \text{and} \quad Q := A'_q = A' \otimes_{Z'} Z'_q.$$

Then

$$(12) \quad Z_p \xrightarrow{\text{Thm 2.12}} R_p \stackrel{(i)}{=} S_q = S'_q \stackrel{(3)}{\cong} Z'_q,$$

where (i) holds since $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$.

Let $e := 1_A - e_{h(\delta)}$. Consider the k -linear map

$$(13) \quad \Psi : P \cong \begin{bmatrix} ePe & ePe_{h(\delta)} \\ e_{h(\delta)}Pe & e_{h(\delta)}Pe_{h(\delta)} \end{bmatrix} \longrightarrow \begin{bmatrix} Q & Q\gamma^* \\ \gamma Q & \gamma Q\gamma^* \end{bmatrix} \subset M_2(\widehat{A}_q)$$

defined using the isomorphism (12) by

$$\begin{bmatrix} a_1 \otimes a_2 & b_1 \otimes b_2 \\ c_1 \otimes c_2 & d_1 \otimes d_2 \end{bmatrix} \mapsto \begin{bmatrix} \psi(a_1) \otimes a_2 & \psi(b_1) \otimes b_2\gamma^* \\ \gamma\psi(c_1) \otimes c_2 & \gamma\psi(d_1) \otimes d_2\gamma^* \end{bmatrix}.$$

By Propositions 2.13 and 2.14, $\delta\delta^* = e_{h(\delta)}$ and $\delta^*\delta = e_{t(\delta)}$. Therefore, since

$$\begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \delta^* \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & \gamma^* \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & e_{h(\delta)} \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & e_{\psi(\delta)} \end{bmatrix},$$

the map Ψ is an algebra homomorphism.

We claim that Ψ is an isomorphism. By Theorem 2.12, ψ is injective on the localized corners ePe , $ePe_{h(\delta)}$, $e_{h(\delta)}Pe$, and $e_{h(\delta)}Pe_{h(\delta)}$. Therefore Ψ is injective. Furthermore, since $\psi : A \rightarrow A'$ is surjective, Ψ surjects onto

$$\begin{bmatrix} A' & A'\gamma^* \\ \gamma A' & \gamma A'\gamma^* \end{bmatrix}.$$

Let $g \in A'_q \setminus A'$. Then $g \in Z'_q$. Thus $g \in Z_p$ by (12). Whence $g \in A_p = P$. Therefore Ψ is surjective, proving our claim.

Consider the bimodules

$${}_{\Psi(P)}M_Q := \begin{bmatrix} Q \\ \gamma Q \end{bmatrix} \quad \text{and} \quad {}_Q N_{\Psi(P)} := \begin{bmatrix} Q & Q\gamma^* \end{bmatrix}.$$

The (Q, Q) -bimodule homomorphism

$$\phi : N \otimes_{\Psi(P)} M \rightarrow Q, \quad \begin{bmatrix} q_1 & q_2\gamma^* \end{bmatrix} \otimes \begin{bmatrix} q_3 \\ \gamma q_4 \end{bmatrix} \mapsto q_1q_2 + q_2\gamma^*\gamma q_4 = q_1q_2 + q_3q_4,$$

and the $(\Psi(P), \Psi(P))$ -bimodule homomorphism

$$\theta : M \otimes_Q N \rightarrow \Psi(P), \quad \begin{bmatrix} q_1 \\ \gamma q_2 \end{bmatrix} \otimes \begin{bmatrix} q_3 & q_4\gamma^* \end{bmatrix} \mapsto \begin{bmatrix} q_1q_3 & q_1q_4\gamma^* \\ \gamma q_2q_3 & \gamma q_2q_4\gamma^* \end{bmatrix},$$

is both surjective. Since P and Q are unital, ϕ and θ are bimodule isomorphisms [C, Lemma 4.5.2]. Therefore by the isomorphism $\Psi(P) \cong P$,

$$(14) \quad P \cong M \otimes_Q N \quad \text{and} \quad Q \cong N \otimes_P M.$$

In general $Q_1^* = \{\delta_1, \dots, \delta_n\}$. A similar argument shows that there are bimodules

$$(15) \quad {}_{\mathbf{P}}M_{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} \\ \gamma_1 \mathbf{Q} \\ \vdots \\ \gamma_n \mathbf{Q} \end{bmatrix} \quad \text{and} \quad {}_{\mathbf{Q}}N_{\mathbf{P}} = \begin{bmatrix} \mathbf{Q} & \mathbf{Q}\gamma_1^* & \cdots & \mathbf{Q}\gamma_n^* \end{bmatrix}.$$

Since $\delta_j^* \delta_j = e_{t(\delta_j)}$ and $\delta_j \delta_j^* = e_{h(\delta_j)}$ for each $1 \leq j \leq n$, (14) holds.

Conversely, suppose $\mathbb{V}(\mathfrak{q}) \cap U = \emptyset$. Then $R_{\mathbf{p}} \neq S_{\mathbf{q}}$. By (3) $S \cong Z'$, whence the centers of \mathbf{P} and \mathbf{Q} are not isomorphic:

$$Z(\mathbf{P}) \cong R_{\mathbf{p}} \neq S_{\mathbf{q}} \cong Z(\mathbf{Q}).$$

Therefore \mathbf{P} and \mathbf{Q} cannot be Morita equivalent. \square

Corollary 2.16. *The noncommutative function fields $A \otimes_Z \text{Frac } Z$ and $A' \otimes_{Z'} \text{Frac } Z'$ are Morita equivalent, and Morita equivalent to $\text{Frac } Z$ and $\text{Frac } Z'$.*

Therefore A , A' , Z , and Z' are noncommutative birationally equivalent.

Proof. We have the following Morita equivalences:

$$A \otimes_Z \text{Frac } Z \stackrel{(i)}{\sim} A' \otimes_{Z'} \text{Frac } Z' \stackrel{(ii)}{\sim} \text{Frac } Z' \stackrel{(iii)}{\sim} \text{Frac } S \stackrel{(iv)}{=} \text{Frac } R.$$

Indeed, (i) follows from Theorem 2.15. (ii) holds since A' is a cancellative and so is a noncommutative crepant resolution, and in particular A' is an endomorphism ring of a finitely generated projective Z' -module. (iii) holds by (3). Finally, (iv) holds since S is a depiction of R by [B2, Theorem 4.15]. \square

Remark 2.17. Consider $\mathfrak{n} \in U$, and set $\mathfrak{m} := \mathfrak{n} \cap R$. Let $\mathfrak{q} \in \text{Spec } S$ be contained in \mathfrak{n} , and set $\mathfrak{p} := \mathfrak{q} \cap R$.

First suppose $V_{\rho'}$ is an A' -module of dimension $1^{Q'_0}$ such that $\mathfrak{n} = \text{ann}_{Z'}(V_{\rho'})$. Then ρ' may be viewed as a vector space diagram on Q' where each arrow is represented by a scalar. By Theorem 2.15, the $A_{\mathbf{p}}$ -module $M \otimes_{A_{\mathbf{q}}} V_{\rho'}$ may be viewed as a vector space diagram $\psi^{-1}\rho'$ on Q of dimension 1^{Q_0} by setting

$$(16) \quad (\psi^{-1}\rho')(a) := \begin{cases} \rho'(\psi(a)) & \text{if } a \in Q_1 \setminus Q_1^* \\ 1 & \text{if } a \in Q_1^* \end{cases}.$$

Now suppose V_{ρ} is an A -module of dimension 1^{Q_0} such that $\mathfrak{m} = \text{ann}_Z(V_{\rho})$. View ρ as a vector space diagram on Q . Then the $A'_{\mathbf{q}}$ -module $N \otimes_{A_{\mathbf{p}}} V_{\rho}$ may be viewed as a vector space diagram $\psi\rho$ on Q' of dimension $1^{Q'_0}$ as follows. By Proposition 2.13, $\rho(\delta) \neq 0$ for each $\delta \in Q_1^*$. Furthermore, no cycle is contracted to a vertex by [B2, Lemma 4.10]. Therefore, up to isomorphism, $\rho(\delta) = 1$ for each $\delta \in Q_1^*$. Thus we may set

$$(\psi\rho)(a) := \rho(\psi^{-1}(a)) \quad \text{for each } a \in Q'_1.$$

It follows that $\rho = \psi^{-1}\rho'$ if and only if $\psi\rho = \rho'$ (which also holds by Theorem 2.15).

3. AZUMAYA LOCI

Throughout, $\psi : A \rightarrow A'$ is a contraction of dimer algebras, with A non-cancellative, A' cancellative, and $S = S'$. Recall that Z and Z' denote the centers of A and A' respectively.

3.1. Localizations that are prime, noetherian, and finite over their centers.

Lemma 3.1. *A is not a finitely generated module over its center.*

Proof. Since A is non-cancellative, there are vertices $i, j \in Q_0$ and a cycle $c \in e_i A e_i$ such that $\bar{\eta}(c) \in \bar{\eta}(e_i A e_i) \setminus \bar{\eta}(e_j A e_j)$. It suffices to suppose that $\sigma \nmid \bar{\eta}(c)$. Let $u \in \mathbb{Z}^2$ be such that $h(c^+) \in \tilde{Q}_0^u$.

Assume to the contrary that $\bar{\eta}(c^n) \in R$ for some $n \geq 2$. Then it suffices to suppose that, modulo I , c^n equals a cycle $p \in e_i A e_i$ that passes through j .

Consider the two paths p_1, p_2 in \tilde{Q} such that $\pi(p_1) = \pi(p_2) = p$, $t(p_1) = t(c^+)$, and $t(p_2) = h(c^+)$. Then p_1 and p_2 must intersect at some vertex $k \in \tilde{Q}_0$.

Consider the subpath s of p_1 from j to k , and the subpath t of p_2 from k to j . Then $d := \pi(ts)$ is a cycle in $e_j A e_j$. Furthermore, $h(d^+)$ is in \tilde{Q}_0^u . Therefore $\bar{\eta}(c) = \bar{\eta}(d)\sigma^m$ for some $m \in \mathbb{Z}$.

But $\sigma = \prod_{D \in \text{Per}_s(A)} x_D$, and by assumption $\sigma \nmid \bar{\eta}(c)$, so $\sigma \nmid \bar{\eta}(c^n) = \bar{\eta}(p)$. Thus $\sigma \nmid \bar{\eta}(d)$. Therefore $\bar{\eta}(c) = \bar{\eta}(d)$. Whence $\bar{\eta}(c) \in \bar{\eta}(e_j A e_j)$, a contradiction.

Thus $c^n \notin Z e_{t(c)}$ for $n \geq 1$ by (3). Therefore A is not module-finite over Z . \square

Proposition 3.2. *Suppose $\mathfrak{q} \in \text{Spec } S$ satisfies $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$, and set $\mathfrak{p} := \mathfrak{q} \cap R$. Then the localized algebra $A_{\mathfrak{p}}$ is prime, noetherian, and a finitely generated module over its center $Z_{\mathfrak{p}}$ with PI degree $|Q_0|$.*

Proof. (i) We first claim that $A_{\mathfrak{p}}$ is noetherian. Since A' is a cancellative dimer algebra, it is noetherian [B, Theorem 2.11]. Thus $A'_{\mathfrak{q}}$ is noetherian. By Theorem 2.15, $A_{\mathfrak{p}}$ is Morita equivalent to $A'_{\mathfrak{q}}$, and therefore $A_{\mathfrak{p}}$ is noetherian as well.

(ii) We now show that $A_{\mathfrak{p}}$ is prime.

Since A' is cancellative, $Z' \cong S$. Since (τ, B) is an impression of A' , the morphism $\text{Max } B \rightarrow \text{Max } Z' = \text{Max } S$ is surjective. Therefore there is an ideal $\mathfrak{b} \in \text{Spec } B$ such that $\mathfrak{b} \cap S = \mathfrak{q}$. In particular, $\mathfrak{b} \cap R = \mathfrak{p}$.

Consider the algebra homomorphism $\eta : A \rightarrow M_{|Q_0|}(B)$ defined in (10). We claim that the induced algebra homomorphism

$$(17) \quad \eta : A_{\mathfrak{p}} \rightarrow M_{|Q_0|}(B_{\mathfrak{b}})$$

is injective. Indeed, since $\bar{\tau} : e_j A' e_i \rightarrow B$ is injective for each $i, j \in Q_0$, the kernel of $\eta : A \rightarrow M_{|Q_0|}(B)$ is generated by elements of the form $p - q$, where $p, q \in A$ are paths such that $p \not\sim q$ while $\psi(p) \sim \psi(q)$.

If either $t(p) \neq t(q)$ or $h(p) \neq h(q)$, then

$$\eta(p) \propto E_{h(p), t(p)} \quad \text{and} \quad \eta(q) \propto E_{h(q), t(q)}$$

have distinct non-zero matrix entries. Therefore $p - q \notin \ker \eta$.

Thus if $p - q \in \ker \eta$ then $t(p) = t(q)$ and $h(p) = h(q)$. But Theorem 2.12 then implies that $p \sim q$ in $A_{\mathfrak{p}}$. Therefore η is an algebra monomorphism on $A_{\mathfrak{p}}$.

$B_{\mathfrak{b}}$ is a domain since B is a polynomial ring. Thus $M_{|Q_0|}(B_{\mathfrak{b}})$ is prime by [L, Proposition 10.20]. Therefore $A_{\mathfrak{p}}$ is prime since η is an algebra monomorphism.

(iii) We now show that $A_{\mathfrak{p}}$ is module-finite over its center $Z_{\mathfrak{p}}$.

We first claim that for each $i, j \in Q_0$,

$$(18) \quad \bar{\eta}(e_i A_{\mathfrak{p}} e_i) = \bar{\eta}(e_j A_{\mathfrak{p}} e_j).$$

Indeed, let $c \in e_i A e_i$ be a cycle. Fix $\mathbf{n} \in \mathbb{V}(\mathbf{q}) \cap U$, and set $\mathbf{m} := \mathbf{n} \cap R$. Let V be a simple A -module of dimension 1^{Q_0} with R -annihilator \mathbf{m} . Then by Proposition 2.9 there are paths $d_1 \in e_j A e_i$ and $d_2 \in e_i A e_j$ such that $d_1, d_2 \notin \text{ann}_A V$ and $d_2 d_1 \in Z_{\mathfrak{p}} e_i$. Let $z \in Z_{\mathfrak{p}}$ be such that $z e_i = d_2 d_1$. Then $z^{-1} \in (Z_{\mathfrak{p}})_{\mathfrak{m}_{\mathfrak{p}}}$. But $(Z_{\mathfrak{p}})_{\mathfrak{m}_{\mathfrak{p}}} = Z_{\mathfrak{p}}$, and so $z^{-1} \in Z_{\mathfrak{p}}$. Thus

$$\bar{\eta}(d_1 c d_2 z^{-1}) = \bar{\eta}(c d_2 d_1 z^{-1}) = \bar{\eta}(c z z^{-1}) = \bar{\eta}(c) \in \bar{\eta}(e_i A_{\mathfrak{p}} e_i).$$

But $d_1 c d_2 z^{-1} \in e_j A_{\mathfrak{p}} e_j$. Therefore $\bar{\eta}(e_i A_{\mathfrak{p}} e_i) \subseteq \bar{\eta}(e_j A_{\mathfrak{p}} e_j)$. Since i, j were arbitrary, this proves (18).

By (ii), η is an algebra monomorphism, and so the restriction $\bar{\eta} : e_i A_{\mathfrak{p}} e_i \rightarrow B_{\mathfrak{b}}$ is an algebra monomorphism for each $i \in Q_0$. Therefore (10) implies that for each $i, j \in Q_0$,

$$e_i A_{\mathfrak{p}} e_i \cong \bar{\eta}(e_i A_{\mathfrak{p}} e_i) = \bar{\eta}(e_j A_{\mathfrak{p}} e_j) \cong e_j A e_j.$$

It follows that $e_i A_{\mathfrak{p}} e_i \subseteq Z_{\mathfrak{p}} e_i$. We may therefore apply the argument [B, second paragraph of proof of Theorem 2.11 with $e_i A e_i = Z e_i$ replaced by $e_i A_{\mathfrak{p}} e_i \subseteq Z_{\mathfrak{p}} e_i$] to conclude that $A_{\mathfrak{p}}$ is module-finite over its center.

(iv) $|Q_0|$ is the PI degree of $A_{\mathfrak{p}}$ by the algebra monomorphism (17) and [B, Lemma 2.4, with $A, U, \tau_{\mathfrak{q}}$ replaced respectively by $A_{\mathfrak{p}}, \{\mathfrak{b}\}, \eta_{\mathfrak{b}}$]. \square

3.2. Azumaya and smooth loci.

Lemma 3.3. *Let $\mathfrak{p} \in \text{Spec } R$. Then*

$$A_{\mathfrak{p}} \otimes_Z R \cong A \otimes_Z R_{\mathfrak{p}}.$$

Proof. It suffices to show that $Z_{\mathfrak{p}} \otimes_Z R \cong R_{\mathfrak{p}}$. Let $z \in Z$ and set $\bar{z} := z + \text{Nil}(Z)$.

If $z^{-1} \in Z_{\mathfrak{p}}$ then $z \notin \mathfrak{p}$. Since $\text{Nil}(Z) \subset \mathfrak{p}$, $z \notin \text{Nil}(Z)$. Thus $\bar{z}^{-1} \in (Z/\text{Nil}(Z))_{\mathfrak{p}}$.

Conversely, if $\bar{z}^{-1} \in (Z/\text{Nil}(Z))_{\mathfrak{p}}$ then $z \notin \text{Nil}(Z) \cup \mathfrak{p}$. Whence $z \notin \mathfrak{p}$. Therefore $z^{-1} \in Z_{\mathfrak{p}}$. \square

For brevity we will write $R_{\mathfrak{m}}/\mathfrak{m}$ and $A_{\mathfrak{m}}/\mathfrak{m}$ for $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ and $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$ respectively.

Lemma 3.4. *Let $\mathfrak{m} \in \text{Max } R$. Then*

$$A \otimes_Z R_{\mathfrak{m}}/\mathfrak{m} \cong A_{\mathfrak{m}}/\mathfrak{m}.$$

Furthermore, if $\mathfrak{q} \in \operatorname{Spec} S$ satisfies $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$ and $\mathfrak{p} = \mathfrak{q} \cap R$, then

$$A \otimes_Z R_{\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

Proof. Consider the short exact sequence

$$(19) \quad 0 \rightarrow \operatorname{Nil}(Z) \rightarrow Z \rightarrow R \cong Z/\operatorname{Nil}(Z) \rightarrow 0.$$

Applying the right exact functor $A_{\mathfrak{m}}/\mathfrak{m} \otimes_Z -$ we obtain the exact sequence

$$A_{\mathfrak{m}}/\mathfrak{m} \otimes_Z \operatorname{Nil}(Z) \rightarrow A_{\mathfrak{m}}/\mathfrak{m} \otimes_Z Z \cong A_{\mathfrak{m}}/\mathfrak{m} \rightarrow A_{\mathfrak{m}}/\mathfrak{m} \otimes_Z R \xrightarrow{\text{Lem 3.3}} A \otimes_Z R_{\mathfrak{m}}/\mathfrak{m} \rightarrow 0.$$

But $\operatorname{Nil}(Z) \subset \mathfrak{m}$, and so the left-most term is zero. Therefore $A \otimes_Z R_{\mathfrak{m}}/\mathfrak{m} \cong A_{\mathfrak{m}}/\mathfrak{m}$.

Now suppose $\mathbb{V}(\mathfrak{q}) \cap U \neq \emptyset$. Applying the right exact functor $A_{\mathfrak{p}} \otimes_Z -$ to (19) we obtain the exact sequence

$$A_{\mathfrak{p}} \otimes_Z \operatorname{Nil}(Z) \rightarrow A_{\mathfrak{p}} \otimes_Z Z \cong A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \otimes_Z R \xrightarrow{\text{Lem 3.3}} A \otimes_Z R_{\mathfrak{p}} \rightarrow 0.$$

But the left-most term is zero by Theorem 2.12. Therefore $A \otimes_Z R_{\mathfrak{p}} \cong A_{\mathfrak{p}}$. \square

Recall that a central simple algebra over k is a simple algebra whose center is k .

Theorem 3.5. *The Azumaya locus \mathcal{A} of A coincides with the intersection of the Azumaya locus \mathcal{A}' of A' and the locus $U \subset \operatorname{Max} Z'$,*

$$\mathcal{A} \cong \mathcal{A}' \cap U.$$

The isomorphism is defined by sending $\mathfrak{n} \in \mathcal{A}' \cap U$ to $\mathfrak{n} \cap R \in \mathcal{A}$.

Proof. Let $\mathfrak{n} \in \operatorname{Max} Z' = \operatorname{Max} S$, and set $\mathfrak{m} := \mathfrak{n} \cap R$. By Remark 2.11, a maximal ideal of Z may be viewed as a maximal ideal of R , and conversely. There are two cases to consider.

(i) First suppose $\mathfrak{n} \notin U$. Then $R_{\mathfrak{m}} \subsetneq S_{\mathfrak{n}}$, so by Lemma ?? $A_{\mathfrak{m}}$ is not module-finite over its center $Z_{\mathfrak{m}}$. Therefore $A_{\mathfrak{m}}$ cannot be Azumaya over its center.

(ii) Now suppose $\mathfrak{n} \in U$. Recall the $(A_{\mathfrak{m}}, A'_{\mathfrak{n}})$ -bimodule M defined in (15). We first claim that

$$A_{\mathfrak{m}}/\mathfrak{m} \cong M \otimes_{A'_{\mathfrak{n}}} A'_{\mathfrak{n}}/\mathfrak{n}.$$

Suppose $Q_1^* = \{\delta_1, \dots, \delta_{\ell}\}$. Then

$$\begin{aligned} A_{\mathfrak{m}}/\mathfrak{m} &\xrightarrow{\text{Lem 3.4}} A \otimes_Z R_{\mathfrak{m}}/\mathfrak{m} \xrightarrow{\text{Lem 3.3}} A_{\mathfrak{m}} \otimes_{Z_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m} \cong \left[\begin{array}{c} \left(1 - \sum_{j=1}^{\ell} e_{h(\delta_j)}\right) A_{\mathfrak{m}} \\ e_{h(\delta_1)} A_{\mathfrak{m}} \\ \vdots \\ e_{h(\delta_{\ell})} A_{\mathfrak{m}} \end{array} \right] \otimes_{Z_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m} \\ &\cong \left[\begin{array}{c} A'_{\mathfrak{n}} \\ \delta_1 A'_{\mathfrak{n}} \\ \vdots \\ \delta_{\ell} A'_{\mathfrak{n}} \end{array} \right] \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m} = M \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m} \stackrel{(i)}{=} M \otimes_{S_{\mathfrak{n}}} S_{\mathfrak{n}}/\mathfrak{n} \cong M \otimes_{A'_{\mathfrak{n}}} A'_{\mathfrak{n}}/\mathfrak{n}, \end{aligned}$$

where (i) holds since $R_{\mathfrak{m}} = S_{\mathfrak{n}}$ is a local ring with unique maximal ideal $\mathfrak{m}R_{\mathfrak{m}} = \mathfrak{n}S_{\mathfrak{n}}$, and Ψ is defined in (13).

Thus, since $M \otimes -$ is a categorical equivalence, $A_{\mathfrak{m}}/\mathfrak{m}$ is central simple over k if and only if $A'_{\mathfrak{n}}/\mathfrak{n}$ is central simple over k . Since $\mathfrak{n} \in U$, $A_{\mathfrak{m}}$ and $A'_{\mathfrak{n}}$ are both prime, noetherian, and module-finite over their centers with PI degrees $|Q_0|$ and $|Q'_0|$ respectively, by Proposition 3.2. Therefore $A_{\mathfrak{m}}$ is Azumaya if and only if $A'_{\mathfrak{n}}$ is Azumaya by the Artin-Processi Theorem [MR, Theorem 13.7.14]. \square

The following corollary gives the first known class of algebras which are not prime, noetherian, or module-finite over their centers, with the property that their Azumaya and smooth loci coincide. The $Y^{p,q}$ dimer algebras are defined in [B, Example 1.3].

Corollary 3.6. *If the Azumaya and smooth loci of A' coincide, then the Azumaya and smooth loci of A coincide. In particular, if A' is a $Y^{p,q}$ algebra then the Azumaya and smooth loci of A coincide.*

Proof. Suppose $\mathfrak{n} \notin U$, and set $\mathfrak{m} := \mathfrak{n} \cap R$. Then $R_{\mathfrak{m}}$ is not finitely generated. Thus the residue field $R_{\mathfrak{m}}/\mathfrak{m}$ has infinite projective dimension over $R_{\mathfrak{m}}$. Therefore \mathfrak{m} is a singular point of R . The corollary then follows from Theorem 3.5. If A' is a $Y^{p,q}$ algebra, then its Azumaya and smooth loci coincide by [B, Theorem 7.3]. \square

The following example shows that \mathcal{A}' and U are distinct in general.

Example 3.7. Let Q and Q' be the quivers shown in Figure 3, with cycles p, q, s in Q . Consider the simple A -module V_{ρ} and simple A' -module $V_{\rho'}$ defined by

$$\rho(a) := \begin{cases} 1 & \text{if } a \text{ is a subpath of } p \\ 0 & \text{otherwise} \end{cases} \quad \text{for } a \in Q_0 \cup Q_1,$$

$$\rho'(a) := \begin{cases} 1 & \text{if } a \text{ is a subpath of } \psi(p) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } a \in Q'_0 \cup Q'_1.$$

Set

$$\mathfrak{m} := \text{ann}_R(V_{\rho}) \in \text{Max } R \quad \text{and} \quad \mathfrak{n} := \text{ann}_{Z'}(V_{\rho'}) \in \text{Max } Z'.$$

Then $\mathfrak{m} = \mathfrak{n} \cap R$ under the isomorphism $Z' \cong S$. We claim that $\mathfrak{n} \in U$ while \mathfrak{n} is not in the Azumaya locus of A' .

To show that $\mathfrak{n} \in U$, it suffices to show that $\bar{\eta}(s) \in R_{\mathfrak{m}}$ since $\bar{\eta}(s)$ is the only irreducible monomial in $S \setminus R$. But $\bar{\eta}(p) \in R \setminus \mathfrak{m}$ since p does not annihilate V_{ρ} . Thus

$$\bar{\eta}(s) = \frac{\bar{\eta}(qs)}{\bar{\eta}(p)} \in R_{\mathfrak{m}}.$$

Therefore $R_{\mathfrak{m}} = S_{\mathfrak{n}}$, whence $\mathfrak{n} \in U$.

Finally, \mathfrak{n} is not in the Azumaya locus of A' since the dimension vector of any simple A' -module of maximal k -dimension is $1^{Q'_0}$ [B, Proposition 2.5, Lemma 2.13].

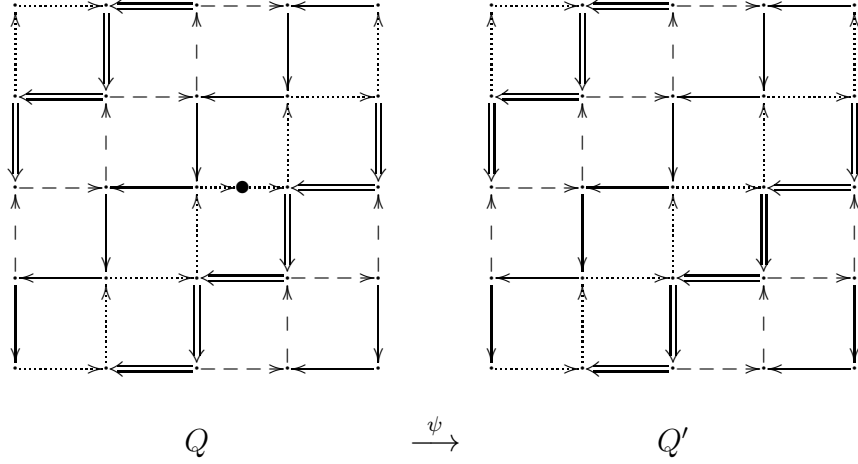


FIGURE 3. The quivers Q and Q' in Example 3.7, drawn on a torus. The cycles p , q , and s in Q are respectively denoted with dotted, dashed, and doubled arrows, with suitably chosen tails.

3.3. Classification of simple modules parameterized by the Azumaya locus.

Recall that a simple A -module V sits over a point \mathfrak{m} in the Azumaya locus \mathcal{A} if $V_{\mathfrak{m}}$ is the unique simple $A_{\mathfrak{m}}$ -module up to isomorphism. The Azumaya locus then parameterizes a family of simple A -module isoclasses.

Proposition 3.8. *$V = V_{\rho}$ is a simple A -module of dimension 1^{Q_0} if and only if V sits over some point $\mathfrak{m} \in \mathcal{A}$.*

Proof. First suppose $\mathfrak{m} \in \mathcal{A}$, and let $V_{\mathfrak{m}}$ be the unique simple $A_{\mathfrak{m}}$ -module. By Proposition 3.2, the PI degree of $A_{\mathfrak{m}}$ is $|Q_0|$. Thus $\dim_k(V_{\mathfrak{m}}) = |Q_0|$.

Denote by \mathfrak{m}_0 the Z -annihilator of any vertex simple A -module; \mathfrak{m}_0 may be viewed as the origin of $\text{Max } Z$. Since \mathfrak{m}_0 is clearly not in \mathcal{A} , $\mathfrak{m} \neq \mathfrak{m}_0$. Thus there is a central element $z \in Z$ such that ze_i is a cycle for each $i \in Q_0$, and $\rho(z) \neq 0$. Since V is simple and z is central, $\rho(z)$ is a scalar multiple of the identity by Schur's lemma. Thus $\rho(ze_i) \neq 0$ for each $i \in Q_0$. Whence $\dim_k(e_i V_{\mathfrak{m}}) \geq 1$, $i \in Q_0$. But $\dim_k(V_{\mathfrak{m}}) = |Q_0|$, and therefore $V_{\mathfrak{m}}$ has dimension 1^{Q_0} .

Now suppose V_{ρ} is simple of dimension 1^{Q_0} . Set $\mathfrak{m} := \text{ann}_A V_{\rho}$. Consider $\rho' := \psi\rho$ as in Remark 2.17. Then $V_{\rho'}$ is simple of dimension $1^{Q'_0}$. Therefore $\mathfrak{n} := \text{ann}_{Z'} V_{\rho'}$ is in \mathcal{A}' .

Furthermore, since V_{ρ} is simple of dimension 1^{Q_0} , there is a cycle t which passes through each vertex and satisfies $\rho(t) \neq 0$. Thus we may apply [B2, proof of Theorem 4.15] to conclude that $R_{\mathfrak{m}} = S_{\mathfrak{n}}$.

Therefore $\mathfrak{n} \in \mathcal{A}' \cap U$. It follows from Theorem 3.5 that $\mathfrak{m} \in \mathcal{A}$. □

In the following, the algebra homomorphism η defined in (10) is used to classify the simple A -modules parameterized by the Azumaya locus. This classification shows that η is very close to being an impression of A even though A may not be embeddable into a matrix ring over a commutative ring; see [B, Proposition 2.5].

Theorem 3.9. *For each A -module V_ρ that sits over a point in the Azumaya locus \mathcal{A} , there is a point $\mathfrak{b} \in \text{Max } B$ such that ρ is isomorphic to the composition*

$$A \xrightarrow{\eta} M_{|Q_0|}(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_{|Q_0|}(B/\mathfrak{b}).$$

Proof. By Proposition 3.8, V_ρ is a simple A -module of dimension 1^{Q_0} . By Remark 2.17, there is a simple A' -module $V_{\rho'}$ of dimension $1^{Q'_0}$ such that $\psi^{-1}\rho' = \rho$. Since (τ, B) is an impression of A , by [B, Proposition 2.5] there is a point $\mathfrak{b} \in \text{Max } B$ such that ρ' is isomorphic to the composition

$$A \xrightarrow{\tau} M_{|Q'_0|}(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_{|Q'_0|}(B/\mathfrak{b}).$$

But then for each $i, j \in Q_0$,

$$\rho|_{e_j A e_i} \cong (\psi^{-1}\rho')|_{e_j A e_i} = \rho'\psi|_{e_j A e_i} \cong \epsilon_{\mathfrak{b}}\tau\psi|_{e_j A e_i} = \epsilon_{\mathfrak{b}}\eta|_{e_j A e_i}.$$

□

3.4. Coordinate-free description of S .

Definition 3.10. Given a quiver algebra $A = kQ/I$ and dimension vector $d = (d_i)_{i \in Q_0}$, denote by $\text{Rep}_d(A)$ the closed affine variety of d -dimensional representations of A viewed as vector space diagrams on Q ,

$$\text{Rep}_d(A) \subset \bigoplus_{a \in Q_1} M_{d_{h(a)} \times d_{t(a)}}(k) = \mathbb{A}_k^{\sum_{a \in Q_1} d_{h(a)} d_{t(a)}}.$$

The reductive algebraic group

$$(20) \quad \text{GL} := \prod_{j \in Q_0} \text{GL}_{d_j}(k)$$

acts linearly on $\text{Rep}_d(A)$ by conjugation (i.e., change-of-basis).

Now let A be a dimer algebra. Denote by $\mathbb{S}(A)$ the open subvariety of $\text{Rep}_{1^{Q_0}}(A)$ of simple representations, and by $\overline{\mathbb{S}(A)}$ its Zariski closure. For an element p in a corner ring $e_j A e_i$, denote by $\mu(p)$ the corresponding function in $k[\text{Rep}_{1^{Q_0}}(A)]$ taking the value $\mu(p)(\rho) := \rho(p) \in k$ on each $\rho \in \text{Rep}_{1^{Q_0}}(A)$.

Proposition 3.11. *Suppose $\psi : A \rightarrow A'$ is a contraction of dimer algebras, with A non-cancellative, A' cancellative, and $S = S'$. Then S and S' are the GL-invariants*

$$S = k[\overline{\mathbb{S}(A)}]^{\text{GL}} \quad \text{and} \quad S' = k[\overline{\mathbb{S}(A')}]^{\text{GL}}.$$

Proof. Let $p, q \in e_j A e_i$ be paths such that $\psi(p) \sim \psi(q)$. By Proposition 2.9, $\mu(p) = \mu(q)$ on $\overline{\mathbb{S}(A)}$. By Theorem 3.9, we may set $\mu(p) = \bar{\eta}(p)$ for each path p in Q . Furthermore, each arrow $a \in Q_1$ vanishes at some point in $\overline{\mathbb{S}(A)}$, and so $\mu(a)$ is not invertible on $\overline{\mathbb{S}(A)}$ (though if $a \in Q_1^\dagger$, then $\mu(a)$ is invertible on $\mathbb{S}(A)$ by Proposition 2.9). Therefore the GL-invariants in $k[\overline{\mathbb{S}(A)}]$ and $k[\overline{\mathbb{S}(A')}]$ are generated by oriented cycles in Q and Q' respectively. \square

4. NON-CANCELLATIVE DIMER ALGEBRAS THAT ARE NOT PI

Cancellative dimer algebras are finitely generated modules over their centers, whereas non-cancellative dimer algebras are not. In this section we show that a class of non-cancellative dimer algebras also do not satisfy a polynomial identity.

S. Paul Smith observed that the non-cancellative dimer algebra with quiver in Figure 4.a is not PI because the quotient $A / \langle ab, ba \rangle$ is isomorphic to the free algebra $k\langle y, z \rangle$. We generalize Smith's result in the following theorem.

Theorem 4.1. *Suppose $\psi : A \rightarrow A'$ is a contraction of dimer algebras with A cancellative, A' non-cancellative, and $S = S'$. Further suppose that for each $\delta \in Q_1^*$ the in-out degree of $h(\delta)$ or $t(\delta)$ is 1. Then A contains a free subalgebra and therefore is not PI.*

Proof. Let p, q be paths in Q from vertex i to vertex j , and without loss of generality suppose that there are arrows $a, b \in Q_1$ such that bap and baq are unit cycles. Then $\psi(a)$ or $\psi(b)$ must be a vertex, and $p \not\sim q$ while $ap \sim aq$ and $pb \sim qb$. In particular, $\bar{\eta}(p) = \bar{\eta}(q)$.

Let r' be a path in Q' from $\psi(j)$ to $\psi(i)$ such that the lift of $r'\psi(p)$ to \tilde{Q}' does not contain a cyclic proper subpath modulo I' . Since τ is injective, Lemma 2.1 implies that $\sigma \nmid \bar{\tau}(r'\psi(p))$.

Since the in-out degree of the head or tail of each contracted arrow is 1, there is a path r in Q from j to i such that $\psi(r) = r'$. Then $\sigma \nmid \bar{\eta}(rp)$ since $\bar{\eta}(rp) = \bar{\tau}(\psi(rp)) = \bar{\tau}(r'\psi(p))$.

Suppose the cycles rp, rq do not generate a free subalgebra of A . By the relations (6), it suffices to suppose that prp equals prq , qrp , or qrq modulo I .

If d is a path such that $dp \sim dq \neq 0$ (resp. $pd \sim qd \neq 0$), then a (resp. b) must be a rightmost (resp. leftmost) arrow subpath of d modulo I . Thus prp must contain a or b modulo I . Therefore since the in-out degree of $h(a) = t(b)$ is 1, prp must contain the path ba modulo I . But then $\sigma \mid \bar{\eta}(prp)$. This yields $\sigma \mid \bar{\eta}(rp)$ since $\sigma = \prod_{D \in \text{Per}_s A} x_D$ and $\bar{\eta}(prp) = \bar{\eta}(p)^2 \bar{\eta}(r)$.

Therefore we have a contradiction, and so $k\langle rp, rq \rangle$ is a free subalgebra of A . \square

Example 4.2. By Proposition 4.1, the dimer algebra with quiver in Figure 4.a contains the free subalgebra $k\langle yz, zy \rangle$. Note that it also contains the free subalgebra $k\langle y, z \rangle$.

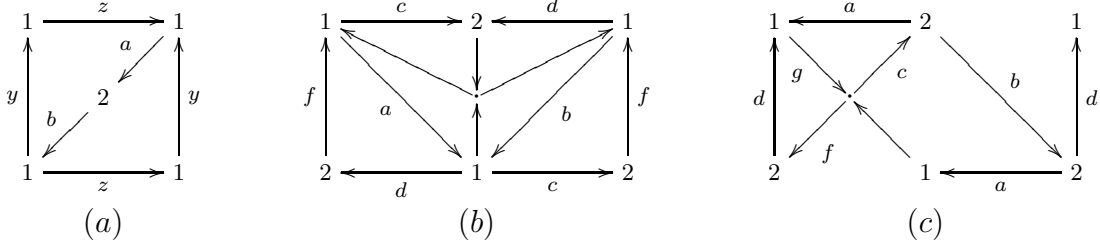


FIGURE 4. The dimer algebras with quivers (a) and (b) are not PI, while it is not known whether the dimer algebra with quiver (c) is PI. All quivers are drawn on a torus and are non-cancellative.

Remark 4.3. The general method of the Proposition 4.1 is to find a pair of non-cancellative paths (the paths p and q in (1)) that are cycles, and determine if these cycles generate a free subalgebra. For example, the dimer algebra with quiver in Figure 4.b contains the free subalgebra $k\langle ab, ba\rangle$, with $p = ab$ and $q = ba$. (Note that it also contains the free subalgebra $k\langle a, b\rangle$.) This algebra is therefore not PI, and also does not satisfy the hypotheses of the proposition.

However, this method will not work in general. For example, the dimer algebra A with quiver in Figure 4.c contains the non-cancellative paths $p = gabbc$ and $q = gdbbf$, which are indeed cycles, but p^2 contains a unit cycle subpath. p and q therefore do not generate a free subalgebra of A , and so it is not known whether this algebra is PI. (See also [B2, Section 4.4, (i.a)] and [DHP, Table 5, 2.3].)

Question 4.4. Is there a non-cancellative dimer algebra that is PI?

APPENDIX A. A BRIEF ACCOUNT OF HIGGSING WITH QUIVERS

Quiver gauge theories

According to string theory, our universe is 10 dimensional.² In many string theories our universe has a product structure $M \times Y$, where M is our usual 4-dimensional space-time and Y is a 6-dimensional compact Calabi-Yau variety.

Let us consider a special class of gauge theories called ‘quiver gauge theories’, which can often be realized in string theory.³ The input for such a theory is a quiver Q , a superpotential W , a dimension vector $d \in \mathbb{N}^{Q_0}$, and a stability parameter $\theta \in \mathbb{R}^{Q_0}$.

Let I be the ideal in $\mathbb{C}Q$ generated by the partial derivatives of W with respect to the arrows in Q . These relations (called ‘F-term relations’) are classical equations of motion from a supersymmetric Lagrangian with superpotential W .⁴ Denote by A the quiver algebra $\mathbb{C}Q/I$.

²More correctly, weakly coupled superstring theory requires 10 dimensions.

³Here we are considering theories with $\mathcal{N} = 1$ supersymmetry.

⁴More correctly, the F-term relations plus the D-term relations imply the equations of motion.

According to these theories, the space X of θ -stable representation isoclasses of dimension d is an affine chart on the compact Calabi-Yau variety Y . The ‘gauge group’ of the theory is the isomorphism group (i.e., change of basis) for representations of A , defined in (20).

Physicists view the elements of A as fields on X . More precisely, A may be viewed as a noncommutative ring of functions on X , where the evaluation of a function $f \in A$ at a point $p \in X$ (i.e., representation p) is the matrix $f(p) := p(f)$ (up to isomorphism).

Vacuum expectation values

Given a path $f \in A$ and a representation $p \in X$, denote by $f(\bar{p})$ the matrix representing f on the vector space diagram on Q associated to p .

The ‘vacuum expectation value’ of a field is its expected (average) energy in the vacuum (think rest mass), and is abbreviated ‘vev’. In our case, the vev of a path $f \in A$ at a point $p \in X$ is the matrix $f(\bar{p})$, which is just the expected energy of f in $M \times \{p\}$.

Higgsing

Spontaneous symmetry breaking is a process where the symmetry of a physical system decreases, and a new property (typically mass) emerges.

For example, suppose a magnet is heated to a high temperature. Then all of its molecules, which are each themselves tiny magnets, jostle and wiggle about randomly. In this heated state the material has rotational symmetry and no net magnet field. However, as the material cools, one molecule happens to settle down first. As the neighboring molecules settle down, they align themselves with the first molecule, until all the molecules settle down in alignment with the first.⁵ The orientation of the first settled molecule then determines the direction of magnetization for the whole material, and the material no longer has rotational symmetry. One says that the rotational symmetry of the heated magnet was spontaneously broken as it cooled, and a global magnetic field emerged.⁶

Higgsing is a way of using spontaneous symmetry breaking to turn a quantum field theory with a massless field and more symmetry into a theory with a massive field and less symmetry. Here mass (vev’s) takes the place of magnetization, gauge symmetry (or the rank of the gauge group) takes the place of rotational symmetry, and energy scale (RG flow) takes the place of temperature.

The recent discovery of the Higgs boson at the Large Hadron Collider is another example of Higgsing.⁷

⁵More precisely there are domains of magnetization.

⁶This is an example of ‘global’ symmetry breaking, meaning the symmetry is physically observable.

⁷This is an example ‘gauge’ symmetry breaking, meaning the symmetry is not an actual observable symmetry of a physical system, but only an artifact of the math used to describe it (like a choice of basis for the matrix of a linear transformation).

Higgsing in quiver gauge theories

Now for our main example. Suppose an arrow a in a quiver gauge theory with dimension 1^{Q_0} is ‘collapsed’ to a vertex e . We make two observations:

- (1) the rank of the gauge group drops by one since the head and tail of a become identified as the single vertex e ;
- (2) a has zero vev at any representation where a is represented by zero, while e can never have zero vev since it is a vertex, and X only consists of representation isoclasses with dimension 1^{Q_0} .

We therefore see that collapsing an arrow to a vertex is a form of Higgsing in quiver gauge theories with dimension 1^{Q_0} .⁸

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⁸This is another example of gauge symmetry breaking.